Prediction of Fully Developed Turbulent Convection with Minimal Explicit Empiricism

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From a purely theoretical analogy between energy and momentum transfer in fully developed turbulent convection, an expression without explicit empiricism, and which appears to be applicable for all channels, all modes of heating on the surface(s), all values of Re in the turbulent regime, and all moderate and large values of Pr, was deduced for Nu. An analogous complementary expression for small values of Pr has the same generality, but incorporates one explicit empirical exponent. The numerical implementation of these two expressions for Nu for specific values of Re and Pr introduces some empiricism, but the resulting uncertainty in Nu, including that associated with the aforementioned exponent, is truly negligible. Because of their accuracy and generality, these two expressions, together with several supplementary relationships, appear to be superior to all existing ones for design calculations. Because of their simple and fundamental structure, they appear to be equally superior for educational purposes.

Introduction

Churchill et al. (2000) consolidated the fragments of the venerable analogy of Reichardt (1951) for fully developed turbulent forced convection in a round tube following a step function in wall temperature into a single expression, and recognized that the latter could be interpreted as an interpolation between the asymptotic solution for $Pr \rightarrow \infty$ and the particular integral solution for $Pr = Pr_t$. They further recognized that this analogy, when explicitly reexpressed in terms of these limiting theoretical solutions, was free of any explicit empiricism, and that the one specific and erroneous empiricism introduced by Reichardt washed out. Because of the absence of any allusion to geometry or to the mode of heating at the wall in the final generic form of the Reichardt analogy, they conjectured that it might have generality for predictions in both of these respects.

Graphical comparisons by Churchill et al. (2000) of this generic expression with the essentially exact numerical solutions of Heng et al. (1998) for a uniformly heated round tube, and with those of Danov et al. (2000) for uniformly and equally heated parallel plates and for parallel plates at different uniform temperatures, confirmed the accuracy and generality of its predictions, as did a comparison with the subsequent more exact numerical solutions of Yu et al. (2001) for both isothermal and uniformly heated round tubes.

The original analogy of Reichardt and its generic interpretation are valid in principle only for $Pr \geq Pr_t \cong 0.87$. However, Churchill et al. (2000) proposed a complementary analogue for lesser values of Pr that is also free of explicit empiricism. This latter expression for $Pr \leq Pr_t$ is equally general, but it has a weaker rationale and lesser accuracy.

These two generic expressions are not only useful for correlation and prediction, their simple algebraic form and their freedom from explicit empiricism provide a radially new insight into the fundamental dependence of turbulent forced convection on Re and Pr. This generality, simplicity, and insight come at a price; in order to obtain a solution in closed form for his differential model, Reichardt necessarily made several simplifications on both mathematical and physical grounds. As a consequence, the original analogy and its generic extension might be expected to be subject to some error both functionally and numerically, despite its freedom from empiricism. The original objective of the work reported herein was to define that error, to evaluate its consequences, and, if appropriate and possible, to reduce it.

The development of the generic correlative and predictive equations of Churchill et al. (2000) depended critically upon a series of prior and rather distinct developments in turbulent flow, in turbulent convection, and in correlation in general. This earlier work is reviewed herein only in such detail as is essential for an understanding of the ensuing analysis.

Mathematical Structure

As noted in the Introduction, the following skeletal structure consists of only those expressions essential to understanding the derivations and constructions that follow.

New structure for turbulent flow in a round tube

Churchill (1997b) expressed the exact time-averaged equation for the conservation of momentum in fully developed turbulent flow of a fluid with invariant physical properties in a round tube in the following new dimensionless form

$$\left(1-\frac{y^+}{a^+}\right)\left[1-\left(\overline{u'v'}\right)^{++}\right]=\frac{du^+}{dy^+} \tag{1}$$

The novelty arises from the introduction of the dimensionless quantity $(\overline{u'v'})^{++} = -\rho\overline{u'v'}/\tau$, which is physically equivalent to the fraction of the transport of momentum in the radial direction due to the turbulent fluctuations. Equation 1 may be integrated formally, following substitution of $R = 1 - y^+/a^+$, to obtain

$$u^+=\frac{a^+}{2}\int_{R^2}^{1}\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^2=\frac{a^+}{2}(1-R^2)-\frac{a^+}{2}\int_{R^2}^{1}\left(\overline{u'v'}\right)^{++}dR^2 \tag{2}$$

It follows, by means of integration by parts, that

$$\left(\frac{2}{f}\right)^{1/2}=u_m^+=\int_0^1 u\,dR^2=\frac{a^+}{4}\int_0^1\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^4=\frac{a^+}{4}-\frac{a^+}{4}\int_0^1\left(\overline{u'v'}\right)^{++}dR^4 \tag{3}$$

The expression of Eq. 1 and thereby Eq. 2 in terms of the quantity $(\overline{u'v'})^{++}$ was the key to the recognition of the possibility of integration by parts, and thereby the attainment of a single integral for u_m^+.

Churchill and Chan (1994) devised a generalized correlating equation for $(\overline{u'v'})^{+} = [1-(y^+/a^+)](\overline{u'v'})^{++}$ that Heng et al. (1998) subsequently updated and reexpressed as

$$\left(\overline{u'v'}\right)^{++}=\left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7}+\left|\exp\left\{\frac{-1}{0.436y^+}\right\}-\frac{1}{0.436a^+}\left(1+\frac{6.95y^+}{a^+}\right)\right|^{-8/7}\right)^{-7/8} \tag{4}$$

The third-power dependence on y^+ in the first term on the righthand side of Eq. 4 was derived by Murphree (1932) and others, while the coefficient of 0.0007 was determined by Rutledge and Sleicher (1993) and others by means of *direct numerical simulations*. The exponential dependence on $(y^+)^{-1}$ in the second term is based on the semi-logarithmic dependence of u^+ on y^+ as derived by Millikan (1938) by speculative dimensional analysis. The structure of the third term on the righthand side was devised by Churchill (1992) to satisfy the condition of $du^+/dy^+ = 0$ at the centerline, as well as to conform to the asymptote $u_c^+ - u^+ \alpha R^2$ for $R \to 0$, which is required to yield a finite value of $(\overline{u'v'})^{++}$ at the centerline. The coefficients of 0.436 and 6.95 were based on the recent precise and extended measurements of Zagarola (1996) for the velocity distribution in the turbulent core at large values of the Reynolds number. The exponent of $-8/7$ was based on the precise and extended measurements of $\overline{u'v'}$ by Wei and Willmarth (1980) for flow between parallel plates. The applicability of these latter data for a round tube hinges on the little-known analogy of MacLeod (1951) for the velocity distributions in these two geometries. The source and basis for a power-mean combination of the terms for large and small values of y^+ is discussed subsequently. The success of a precursor of Eq. 4 in representing the experimental data of Wei and Willmarth was demonstrated graphically by Churchill and Chan (1994). These details concerning Eq. 4 are presented here because all of the subsequent expressions depend critically on its accuracy, functionality, and generality, and because each of the cited contributions was essential to its construction.

The computed values of Yu et al. (2000) based on Eqs. 3 and 4 may be closely represented by

$$\left(\frac{2}{f}\right)^{1/2}=u_m^+=3.2-\frac{227}{a^+}+\left(\frac{50}{a^+}\right)^2+\frac{1}{0.436}\ln\{a^+ \tag{5}$$

The form of Eq. 5, including the unfamiliar terms in $(a^+)^{-1}$ and $(a+)^{-2}$ directly follow from the form of Eq. 4, as do the coefficients.

New structure for turbulent convection

Churchill (1997b) proposed the following analogue of Eq. 1 for fully developed turbulent convection in a round tube with invariant physical properties and negligible viscous dissipation

$$(1+\gamma)\left(1-\frac{y^+}{a^+}\right)\left[1-\left(\overline{T'v'}\right)^{++}\right]=\frac{dT^+}{dy^+} \tag{6}$$

where

$$1+\gamma=\frac{1}{R^2}\int_0^{R^2}\frac{u^+}{u_m^+}\left(\frac{\partial T^+/\partial x}{\partial T_m^+/\partial x}\right)dR^2 \tag{7}$$

The primary novelty here is the direct use of $(\overline{T'v'})^{++} = \rho c_p\overline{T'v'}/j$, which is physically equivalent to the fraction of the transport of energy in the y-direction due to the turbulent fluctuations, but the improvement is also due to inclusion of the quantity γ, which represents the fractional deviation of the radial heat flux density distribution from the radial shear stress distribution. This quantity has been neglected unjustifiably (set equal to zero) in many prior formulations and calculations.

Substituting, because of its more constrained variance, Pr_t/Pr for $(\overline{T'v'})^{++}$ by virtue of the relationship

$$\frac{Pr_t}{Pr} = \frac{(\overline{u'v'})^{++}\left[1-(\overline{T'v'})^{++}\right]}{(\overline{T'v'})^{++}\left[1-(\overline{u'v'})^{++}\right]} \tag{8}$$

converts Eq. 6 to

$$\frac{(1+\gamma)\left(1-\dfrac{y^+}{a^+}\right)}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)} = \frac{dT^+}{dy^+} \tag{9}$$

On the basis of Eq. 8, Pr_t/Pr may be noted to represent the ratio of the transport of momentum by the turbulent fluctuations to that by the molecular motions (the viscous shear), divided by the equivalent ratio for the transport of energy, and to be independent of its original heuristic, diffusional basis. The designation Pr_t/Pr was retained for historical reasons, despite this misleading implication.

The formal integration of Eq. 9 yields the following integral expression for the temperature distribution across the radius of the tube

$$T^+ = \frac{a^+}{2}\int_{R^2}^{1}\frac{(1+\gamma)dR^2}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)} \tag{10}$$

Equations 6, 7, 9, and 10 are exact within the constraints noted with respect to Eq. 6, and Eq. 8 is simply a definition of the quantity Pr_t/Pr in terms of well-defined physical quantities.

At this point, it is expeditious to proceed separately and sequentially for two particular thermal boundary conditions.

Uniform Heating. For the imposition of a uniform heat flux density from the tube wall to the fluid beginning at some particular length x_0, $\partial T^+/\partial x \to \partial T_m^+/\partial x$ as x increases. Hence, for fully developed convection, Eq. 7 may be reduced to

$$1+\gamma = \frac{1}{R^2}\int_0^{R^2}\left(\frac{u^+}{u_m^+}\right)dR^2 \tag{11}$$

By virtue of Eqs. 2 and 3, and integration by parts, Eq. 11 may be reexpressed exactly in terms of a combination of several single integrals of $(\overline{u'v'})^{++}$ (see Churchill (1997b) or Heng et al. (1998)). Similarly, by virtue of Eq. 11 and integration by parts, the integration of T^+, weighted by u^+/u_m^+, may be expressed as

$$\frac{2a^+}{Nu} = T_m^+ = \frac{a^+}{4}\int_0^1\frac{(1+\gamma)^2 dR^4}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)} \tag{12}$$

Equation 12 is also exact within the constraints imposed on Eq. 6.

The dependence of Pr_t on Pr, y^+, and a^+, or possibly on Pr and $(\overline{u'v'})^{++}$ only, is not yet known with certainty. However, this uncertainty may be avoided without question for one particular condition, and speculatively for two others. The corresponding three expressions for Nu were formulated as follows with the expectation that they might prove useful for interpretation and correlation.

For $Pr = 0$, Eq. 12 unambiguously reduces to

$$Nu_0 \equiv Nu\{Pr=0\} = \frac{8}{\int(1+\gamma)^2 dR^4} = \frac{8}{(1+\gamma)^2_{mR^4}} \tag{13}$$

Insofar as $Pr_t = Pr$ for all y^+ and all a^+ for some particular value of Pr, Eq. 12 may be reduced to

$$\begin{aligned} Nu_1 \equiv Nu\{Pr_t = Pr\} &= \frac{8}{\int_0^1(1+\gamma)^2\left[1-(\overline{u'v'})^{++}\right]dR^4} \\ &= \frac{2a^+}{(1+\gamma)^2_{wmR^4}u_m^+} = \frac{Re\,f/2}{(1+\gamma)^2_{wmR^4}} \end{aligned} \tag{14}$$

The right-most forms of Eq. 14 follow from Eq. 3. The subscript wmR^4 designates the integrated mean value of $(1+\gamma)^2$, weighted by $1-(\overline{u'v'})^{++}$, over R^4.

Insofar as Pr_t approaches a fixed value as $y^+ \to 0$ for large values of Pr, Eq. 10 may be integrated exactly, using $(\overline{u'v'})^{++} = 0.7(y^+/10)^3$, to obtain

$$Nu_\infty \equiv Nu\{Pr\to\infty\} = \frac{3^{3/2}(0.7)^{1/3}}{20\pi} \times\left(1-\frac{Pr_t}{Pr}\right)^{4/3}\left(\frac{Pr}{Pr_t}\right)^{1/3}\left(\frac{T_c^+}{T_m^+}\right)Re\left(\frac{f}{2}\right)^{1/2} \tag{15a}$$

$$\Rightarrow 0.07343\left(\frac{Pr}{Pr_t}\right)^{1/3}\left(\frac{T_c^+}{T_m^+}\right)Re\left(\frac{f}{2}\right)^{1/2} \tag{15b}$$

$$\Rightarrow 0.07343\left(\frac{Pr}{Pr_t}\right)^{1/3}Re\left(\frac{f}{2}\right)^{1/2} \tag{15c}$$

Equation 15a is an exact upper bounding asymptote, Eq. 15b is a lower bounding asymptote with a more limited range, and Eq. 15c is the limiting asymptote. The expanded form (Eq. 15a) was apparently first derived by Churchill (1992, 1997a), but the final form (Eq. 15c) was obtained earlier by Petukhov (1970) and others.

Uniform Wall Temperature. For a step in wall temperature at some distance x_0, $(\partial T^+/\partial x)/(\partial T_m^+/\partial x) \to T^+/T_m^+$ as x increases beyond x_0, and Eq. 7 reduces to

$$1+\gamma = \frac{1}{R^2}\int_0^{R^2}\frac{u^+}{u_m^+}\left(\frac{T^+}{T_m^+}\right)dR^2 \tag{16}$$

With this expression for γ, which depends on T^+ as well as on u^+, the double integral for T_m^+ cannot be reduced to a single one by integration by parts, and Eqs. 10 and 16 must be solved iteratively. Numerically, a better method is to solve the differential counterparts of Eqs. 10 and 16, as well as Eq. 1, stepwise and simultaneously for trial values of T_m^+.

An alternative formulation for Nu, which avoids a double integral, may be developed in terms of T_c^+ rather than T_m^+ as follows. From Eq. 10 for $R = 1$

$$T_c^+ = \frac{2a^+}{Nu}\left(\frac{T_c^+}{T_m^+}\right) = \frac{a^+}{2}\int_0^1 \frac{(1+\gamma)dR^2}{1+\dfrac{Pr}{Pr_t}\left(\dfrac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)} \tag{17}$$

The reduced formulation for $Pr = 0$ now becomes

$$Nu_0 = \frac{4(T_c^+/T_m^+)_0}{\int(1+\gamma)dR^2} = \frac{4(T_c^+/T_m^+)_0}{(1+\gamma)_{mR^2}} \tag{18}$$

and that for $Pr = Pr_t$ becomes

$$Nu_1 = \frac{4(T_c^+/T_m^+)_1}{\int_0^1(1+\gamma)\left[1-\left(\overline{u'v'}\right)^{++}\right]dR^2} = \frac{u_m^+}{u_c^+}\left(\frac{T_c^+}{T_m^+}\right)_1 \frac{Re\,f/2}{(1+\gamma)_{wmR^2}} \tag{19}$$

Here the subscript wmR^2 designates the integrated mean value of $1+\gamma$, weighted by $1-(\overline{u'v'})^{++}$, over R^2. Equation 15 remains applicable for an isothermal tube wall. Equations 17–19, which are applicable for uniform heating, as well as for uniform wall temperature, are useful for interpretation and correlation, but not for numerical evaluations for the latter condition since T_m^+ must then still be evaluated separately.

Other geometries and modes of heating

Equations 1 and 6 are directly applicable for fully developed flow and convection, respectively, in all geometries if $1-(y^+/a^+)$ is simply replaced by the more general expression τ/τ_w. For parallel-plate channels, Eqs. 1, 2, 6, 10, 17, and 18 are directly applicable if a^+ is simply replaced by b^+, and R is correspondingly interpreted as equal to $1-(y^+/b^+)$. Insofar as the analogy of MacLeod is valid, Eq. 4 is applicable as well. The expressions for u_m^+, γ, T_m^+ and, in general, Nu differ from geometry to geometry, but Eqs. 8 and 15 are universally applicable. The differing expressions are readily derived simply by accounting for geometry, and, therefore, will not be presented here. The details for parallel plates and two modes of heating may be found in Danov et al. (2000).

Uniqueness

Most of the above expressions for flow and convection differ from the conventional ones because of their formulation in terms of $(\overline{u'v'})^{++}$. In principle, all of them could have been derived in terms of the eddy viscosity ratio μ_t/μ, which for round tubes and parallel plates is exactly equivalent to $(\overline{u'v'})^{++}/[1-(\overline{u'v'})^{++}]$. However, because of the greater complexity of the expressions in terms of μ_t/μ, some of the reductions presented herein apparently were never recognized.

Generic Correlating Equations

Churchill and Usagi (1972) proposed the expression

$$(y\{x\})^p = (y_0\{x\})^p + (y_\infty\{x\})^p \tag{20}$$

as a generic correlating equation for interpolation between two asymptotes or limiting values. (Equation 4 is an application of Eq. 20 with $p = -8/7$.) For three regimes, Eq. 20 may simply be applied twice in series, for example, as

$$y^p = [y_0^q + y_i^q]^{p/q} + y_\infty^p \tag{21}$$

In Eq. 21 and all subsequent expressions of this type, the dependence on x is, in the interests of simplicity, merely implied. In some instances, depending on the functionality of y_0, y_i, and y_∞, Eq. 21 may predict forbidden values of y, that is, either below the lower bounding value or asymptote or above the upper one. The magnitude of the resulting error may be in some instances so small as to be tolerable. In any event, such an anomaly may be avoided with staggered formulations such as

$$y^p - y_0^p = \left[y_i^{pq} + (y_\infty^p - y_0^p)^q\right]^{1/q} \tag{22}$$

although other anomalies may occur, again depending on the particular, relative functionalities of y_0, y_i, and y_∞. The elimination of the possible flaw in Eq. 21 by Eq. 22 comes at the price of greater complexity, even though the same asymptotes and the same number of arbitrary exponents are involved. In the interests of brevity the alternative expressions that result from the reverse order of combination of y_0, y_i, and y_∞ in Eqs. 21 and 22 will not be presented here, even though they have equal potentiality for correlation and should always be tested in a specific application to determine the most successful representation.

This diversion on generic correlation equations was inserted here, since Eq. 21 is applied and Eq. 22 proves to be a critical element in the developments that follow.

Reichardt Analogy

A simple satisfactory correlating equation in terms of Nu_0, Nu_1, and Nu_∞ and with the structure of Eq. 21 or Eq. 22 was not found directly. Instead, the analogy of Reichardt (1951) was discovered somewhat serendipitously to fulfill this role. In the interests of consistency and simplicity, the original derivation of the analogy by Reichardt is rephrased in terms of $(\overline{u'v'})^{++}$, rather than μ_t/μ. Taking the ratio of Eqs. 1 and 9, followed by formal integration results in

$$T_c^+ = \int_0^{u_c^+} \frac{(1+\gamma)du^+}{1+\left(\dfrac{Pr}{Pr_t}-1\right)\left(\overline{u'v'}\right)^{++}} \tag{23}$$

The ingenious, critical step by Reinchardt that permitted the derivation of a solution in closed form, starting from Eq. 23, was the exact expansion of the integrand into three additive terms as follows

$$T_c^+ = \int_0^{u_c^+} \left[\frac{\gamma}{1+\left(\frac{Pr}{Pr_t}-1\right)\left(\overline{u'v'}\right)^{++}} + \frac{Pr_t}{Pr} + \frac{1-\frac{Pr_t}{Pr}}{1+\frac{Pr}{Pr_t}\left(\frac{\left(\overline{u'v'}\right)^{++}}{1-\left(\overline{u'v'}\right)^{++}}\right)} \right] du^+ \tag{24}$$

He next made four approximations: (1) the representation of the left-most term of the integrand by $\gamma Pr_t/Pr$; (2) the representation of $(\overline{u'v'})^{++}/[1-(\overline{u'v'})^{++}]$ in the right-most term by $2.7(y^+/10)^5$; (3) the replacement of du^+ by dy^+ in the right-most term; and (4) the neglect of the variation of Pr_t with y^+. The quantitative evaluation of each of these four approximations was one of the original objectives of this investigation.

After all of these preparations, Reichardt did not actually derive a final single algebraic expression for Nu, but instead used fragments thereof to compute numerical values of Nu, most of which he only presented graphically. Had he derived such an expression and used $0.7\ (y^+/10)^3$, rather than $2.7\,(y^+/10)^5$ in the righthand term, he presumably would have obtained

$$\frac{1}{Nu} = \frac{Pr_t}{Pr}\left(\frac{T_m^+}{T_c^+}\right)\left(\frac{u_c^+}{u_m^+}\right)\frac{(1+\gamma)_{mu^+}}{Re\,f/2} + \left(1-\frac{Pr_t}{Pr}\right)\left(\frac{T_m^+}{T_c^+}\right)\left(\frac{Pr_t}{Pr}\right)^{1/3}\frac{13.62}{Re(f/2)^{1/2}} \tag{25}$$

Because of the prior derivation of Eq. 15b for Nu_∞ and Eq. 19 for Nu_1 for a uniform wall temperature, Churchill et al. (2000) recognized that Eq. 25 could be interpreted as

$$\frac{1}{Nu} = \left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left(1-\frac{Pr_t}{Pr}\right)\frac{1}{Nu_\infty} \tag{26}$$

if T_c^+ and T_m^+ were evaluated at their limiting values. Test calculations using Eq. 25 with local values of T_c^+/T_m^+ did not significantly differ from those using Eq. 26, which, on the basis of Eqs. 15b and 19, incorporates the local values. Reichardt referred only offhandedly in his derivation to the mode of heating, but utilized Eq. 16 rather than Eq. 11 for γ, which is the key difference. On the basis of a footnote regarding the contemporaneous solution of Lyon (1951), it may be inferred that Reichardt did not fully recognize the reason for the discrepancy between the solution of Lyon for uniform heating and his own solution for uniform wall temperature. This distinction was first stressed by Seban and Shimazaki (1951) later that same year. On the other hand, the absence of any intimation of the geometry or the mode of heating in Eq. 26 led Churchill et al. (2000) to speculate that it might be broadly applicable in both of these senses.

Equation 26 may be interpreted to have the form of Eq. 20 with $p = -1$ and with $Nu_1(Pr/Pr_t)$ and $Nu_\infty/(1-Pr_t/Pr)$ as asymptotes. The first of these terms might have been conceived in advance as an asymptote, but hardly the latter. On the other hand, the rearrangement of Eq. 26 as

$$\frac{Nu-Nu_1}{Nu_\infty-Nu_1} = \frac{1}{1+\frac{Nu_\infty}{Nu_1}\left(\frac{Pr_t}{Pr-Pr_t}\right)} \tag{27}$$

may be recognized to have the form of Eq. 22 with $p = -q = 1$, $y_0 = Nu_1$, $y_\infty = Nu_\infty$, and $y_i = (Nu_1/Nu_\infty)(Nu_\infty - Nu_1)[(Pr/Pr_t)-1])$. In terms of the staggered variable $Pr/Pr_t - 1$, Nu goes through a classical sigmoidal transition from Nu_1, to Nu_∞ with an intermediate asymptote given by $Nu_i = (Nu_1/Nu_\infty)(Nu_\infty - Nu_1)([(Pr/Pr_t)-1])$. The effect of the staggered variable $(Pr/Pr_t)-1$ is to convert Nu_1 from a particular value to a lower bounding asymptote. This functionality would certainly not have been deciphered from experimental data or even from the computed values of Yu et al. (2001), and probably not even from Eq. 26, in the absence of Eq. 22.

Analogue of Reichardt Analogy for $Pr \leq Pr_t$

It is apparent from Eqs. 25 and 26 that their applicability is limited to $Pr \geq Pr_t$. Churchill et al. (2000) conjectured that the complete inverse of Eq. 26, namely

$$\frac{1}{Nu} = \left(\frac{Pr}{Pr_t}\right)\frac{1}{Nu_1} + \left(1-\frac{Pr}{Pr_t}\right)\frac{1}{Nu_0} \tag{28}$$

might provide a good prediction for $Pr \leq Pr_t$. However, Eq. 28 proved to be quite numerically inadequate, and furthermore resulted in a discrete step in $dNu/d(Pr/Pr_t)$ at $Pr = Pr_t$. Accordingly, they investigated the representation provided by various modified expressions, including, most successfully

$$\frac{Nu-Nu_0}{Nu_1-Nu_0} = \frac{1}{1+\alpha\left(\frac{Nu_1}{Nu_0}\right)\left(\frac{Pr_t-Pr}{Pr}\right)} \tag{29}$$

where α is an arbitrary coefficient, for which they derived an expression by equating the derivatives of Eqs. 27 and 29 at $Pr = Pr_t$ and, thereby, obtained

$$\frac{Nu-Nu_0}{Nu_1-Nu_0} = \frac{1}{1+\frac{Nu_1}{Nu_\infty^1}\left(\frac{Nu_\infty^1-Nu_1}{Nu_1-Nu_0}\right)\left(\frac{Pr_t-Pr}{Pr}\right)} \tag{30}$$

where $Nu_\infty^1 = Nu_\infty\{Pr = Pr_t\}$. Remarkably, Eq. 30 is free of any explicit empiricism and might be expected to have the same generality as Eq. 27, despite its lesser theoretical credentials. The functional behavior predicted by Eq. 30 is wholly analogous to that predicted by Eq. 27, namely a sigmoidal transition from Nu_0 to Nu_1, with the effective internal asymptote

$$Nu_i = \frac{Nu_\infty^1}{Nu_1} \frac{(Nu_1 - Nu_0)^2}{(Nu_\infty^1 - Nu_1)} \left(\frac{Pr_t - Pr}{Pr} \right) \tag{31}$$

Again, by virtue of a staggered independent variable, this time $(Pr/Pr) - 1$, Nu_1 becomes a limiting rather than a particular value.

The combination of Eqs. 27 and 30 was expected to provide a good functional and possibly numerical prediction for Nu for all values of Pr/Pr_t, all values of a^+ or the equivalent, greater than 150, all geometries, and all modes of heating at the surface(s), without any explicit empiricism. Equation 26, and thereby Eq. 27, is, of course, not exact owing to the approximations made by Reichardt in order to be able to integrate analytically, and Eq. 30 would be expected to be subject to even greater error due to its purely conjectural derivation. The actual resulting functional and numerical errors are examined in the next section.

Intrinsic Errors in the Generalized Reichardt Analogy

A comparison of the predictions of Nu by the serial combination of Eqs. 27 and 30 with the essentially exact computed values of Yu et al. (2001) for a round tube with a uniform wall temperature is reproduced in Figure 1. The agreement appears to be very good both functionally and numerically for all values of Pr/Pr_t, and for all three of the chosen representative values of a^+. Equivalent representations were demonstrated by Yu et al. for a uniformly heated round tube and by Churchill et al. (2000) for a parallel-plate channel with both equal uniform heating and unequal uniform wall temperatures. These comparisons are independent of the expression used for Pr_t in the numerical calculations for intermediate values of Pr insofar as the condition leading to Eqs. 14 and 19, namely, independence of Pr_t from y^+ for $Pr = Pr_t$, and

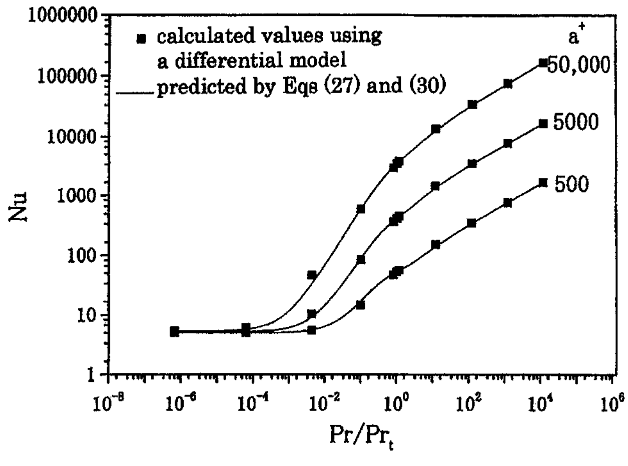

Figure 1. Predictions of *Nu* by Eqs. 27 and 30 vs. calculated values of Yu et al. (2001) for round tubes with a uniform wall temperature.

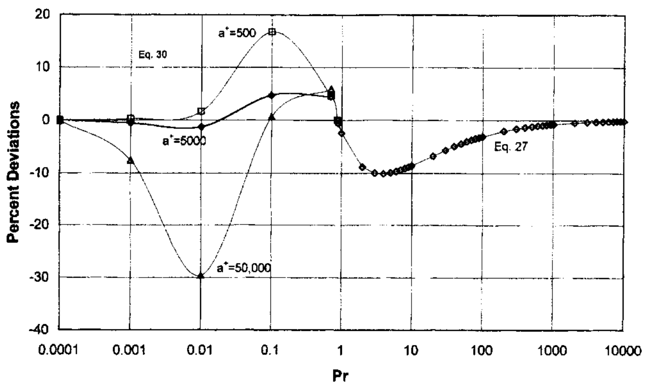

Figure 2. Percent deviations of predictions of Eqs. 27 and 30 from calculated values of Yu et al. (2001) for round tubes with a uniform wall temperature.

the condition leading to all versions of Eq. 15 (namely, the approach to a finite value of Pr_t as $y^+ \to 0$ for large values of Pr) are fulfilled.

A more critical test is provided in Figure 2 in which the deviations of the predictions of Eqs. 27 and 30 from the calculated values of Yu et al. are plotted as percentages in semi-logarithmic coordinates. Both the predicted and computed values in Figure 2 are arbitrarily based on the following empirical expression for Pr

$$Pr_t = 0.85 + \frac{0.015}{Pr} \tag{32}$$

Any alternative expression for Pr_t that is also a function only of Pr would simply shift the curves and the computed values in Figure 2 horizontally and equally.

In contrast to the impression of good accuracy provided by Figure 1, the percentage of deviations in Figure 2 may be seen to be significant in magnitude. The three curves on the lefthand side (for $Pr < Pr_t = 0.8673$) represent Eq. 30 for $a^+ = 500$, 5,000, and 50,000. The pattern of the deviations appears to be quite irregular with respect to both Pr and a^+. On the other hand, the deviations on the righthand side (for $Pr > Pr_t = 0.8673$) are more constrained and essentially are independent of a^+. For that reason, a curve representing Eq. 27 was plotted only for $a^+ = 5{,}000$. The pattern of the deviations from Eqs. 27 and 30 was found to be very nearly the same for a uniformly heated round rube and for parallel plates with equal uniform heating and unequal uniform temperatures, implying a fundamental shortcoming of Eqs. 27 and 30.

A numerical assessment of the approximations made by Reichardt in simplifying the integration of Eq. 24 revealed that those made with respect to the right-most term were primarily responsible for the error indicated in Figure 2 for $Pr > Pr_t = 0.8673$. A much more detailed analytical solution was derived with no approximations other than utilizing $0.7(y^+/10)^3$ for $(\overline{u'v'})^{++}$. However, the result was very disappointing in that the negative deviations were replaced by positive ones of almost the same magnitude, apparently due to the inapplicability of this approximation for $(\overline{u'v'})^{++}$ for $y^+ > 5$. At this point, a completely different and much more successful approach was discovered.

Alternative Analogy

Churchill (1997a) devised, on the basis of a generalized temperature distribution and μ_t/μ, an analogy that may now be derived more straighforwardly and in terms of $\overline{(u'v')}^{++}$ as follows. For "the turbulent core near the wall" ($30 < y^+ < 0.1a^+$), the general semi-logarithmic expression derived by Millikan (1938) for the time-mean velocity distribution is known from the recent experimental measurements of Zagarola (1996) to provide a good representation. It follows that in this region $1 - y^+/a^+ \cong 1$ and $\overline{(u'v')}^{++} \cong 1 - 1/0.436y^+$. For purposes of simplicity only, γ will be postulated to be zero at this stage and then accounted for at the end. Then Eq. 9 reduces to

$$\frac{dT^+}{dy^+} = \frac{1}{1 + \frac{Pr}{Pr_t}(0.436y^+ - 1)} \tag{33}$$

Utilizing indefinite integration, since Eq. 33 is not applicable at $y^+ = 0$, and postulating the independence of Pr_t from y^+ results in

$$T^+ = \frac{1}{0.436}\left(\frac{Pr_t}{Pr}\right)\ln\left\{y^+ - \frac{1}{0.436}\left(1 - \frac{Pr_t}{Pr}\right)\right\} + \Psi\left\{\frac{Pr_t}{Pr}\right\} \tag{34}$$

Then

$$T_c^+ = \frac{1}{0.436}\left(\frac{Pr_t}{Pr}\right)\ln\left\{a^+ - \frac{1}{0.436}\left(1 - \frac{Pr_t}{Pr}\right)\right\} + \Psi\left\{\frac{Pr_t}{Pr}\right\} \tag{35}$$

For all applicable values of a^+, that is, those greater than 150, and all values of $Pr > Pr_t$, Eq. 35 may be reduced with negligible error to

$$T_c^+ = \frac{1}{0.436}\left(\frac{Pr_t}{Pr}\right)\ln\{a^+\} + \Psi\left\{\frac{Pr_t}{Pr}\right\} \tag{36}$$

The measurements of the time-mean velocity at the centerline by Zagarola may be very closely represented by

$$u_c^+ = 7.64 + \frac{1}{0.436}\ln\{a^+\} \tag{37}$$

Hence, Eqs. 36 and 37 may be combined to obtain

$$T_c^+ = \frac{Pr_t}{Pr}(u_c^+ - 7.64) + \Psi\left\{\frac{Pr_t}{Pr}\right\} \tag{38}$$

It follows that

$$Nu = \frac{2a^+}{T_m^+} = \frac{2a^+}{T_c^+}\left(\frac{T_c^+}{T_m^+}\right) = \frac{2a^+(T_c^+/T_m^+)}{\frac{Pr_t}{Pr}(u_c^+ - 7.64) + \Psi\left\{\frac{Pr_t}{Pr}\right\}} \tag{39}$$

Equation 39 reduces to Nu_1 (for $\gamma = 0$) and Nu_∞ only if

$$\Psi\left\{\frac{Pr_t}{Pr}\right\} = \frac{Pr_t}{Pr}\left[13.62\left(\frac{Pr_t}{Pr}\right)^{2/3} - 5.98\right] \tag{40}$$

Substitution for $\Psi\{Pr_t/Pr\}$ in Eq. 39 from Eq. 40 results in the equivalent of

$$Nu = \frac{1}{\left(\frac{Pr_t}{Pr}\right)\frac{1}{Nu_1} + \left[1 - \left(\frac{Pr_t}{Pr}\right)^{2/3}\right]\frac{1}{Nu_\infty}} \tag{41}$$

with Nu_∞ from Eq. 15c and Nu_1 from Eq. 19 (with $\gamma = 0$, and T_c^+/T_m^+ allowed to vary). It may be inferred that Eq. 41 remains applicable for the mean values of γ indicated by Eqs. 14 and 19 for uniform heating and isothermal heating, respectively. Equation 41, just as Eq. 26, is free of explicit empiricism.

Equation 41 predicts the computed values of Yu et al. (2001) for uniform wall temperature, all $a^+ \geq 150$, and $Pr \geq Pr_t$ with a positive error of about 0.25% on the mean, and with a maximum error of 0.7%. Thus, the term $1-(Pr_t/Pr)^{2/3}$ in Eq. 41, as compared to $1-(Pr_t/Pr)$ in Eq. 26, almost exactly compensates for the error due to the approximations of Reichardt. The term in $1-(Pr_t/Pr)^{2/3}$ was apparently first utilized by Ribaud (1941) in a correlating equation, then subsequently by Petukhov and Popov (1963), both on purely empirical grounds.

Many expressions for $Pr < Pr_t$, complementary to Eq. 41, were examined, the most successful being

$$\frac{Nu_1 - Nu}{Nu_1 - Nu_0} = \frac{1}{1 + \frac{(Pr_t/Pr)^{1/8}(Nu_1 - Nu_0)Nu_\infty^1}{\left(\frac{Pr_t}{Pr} - 1\right)\left(Nu_\infty^1 - \frac{2}{3}Nu_1\right)Nu_1}} \tag{42}$$

Equation 42 is loosely analogous to Eq. 41 and has a matching value and derivative with respect to Pr/Pr_t at $Pr = Pr_t$, but the exponent of 1/8 is purely empirical. Equation 42 represents the computed values of Nu for $Pr \leq Pr_t$ and $a^+ \geq 150$ for uniform wall temperature, uniform heating, and parallel plate channels with an error of less than 1% on the mean and a maximum of 3%. The predictions of Eqs. 41 and 42 are subsequently compared graphically with the numerically computed values for uniform wall temperature, as well as with prior correlative and predictive expressions.

Numerical Implementation

Although Eq. 41 is free of explicit empiricism and Eq. 42 involves only a minimal degree numerically, their implementation in practice, that is, the prediction of numerical values of Nu for specific values of Re and Pr, involves a considerable degree of empiricism, but very little associated uncertainty. This aspect is described here only for a round tube with isothermal heating since the development of equivalent supplemental expressions for other geometries and modes of heating is straightforward.

Yu et al. (2001) carried out calculations for Nu and T_c^+/T_m, and tabulated these values for a series of values of a^+ and Pr. For tabulated values of a^+, but intermediate values of Pr, Eq. 32 may be used for Pr_t and then Eqs. 41 and 42 for Nu, since $Nu_\infty = 0.07343(Pr/Pr_t)^{1/3}(2a^+)$ and values of Nu_0 and Nu_1 are included in the tabulation. For intermediate values of a^+, it might appear necessary to have separate correlating equations for u_c^+/u_m^+, T_c^+/T_m^+, $(1+\gamma)_{mR^2}$ and $(1+\gamma)_{wmR^2}$. However, Yu et al. simplified the representation by developing the following purely empirical expressions for the direct prediction of Nu_0 and Nu_1

$$Nu_0 = \frac{4(T_c^+/T_m^+)}{(1+\gamma)_{mR^2}} \cong \frac{8}{1+\dfrac{1.54}{(u_m^+)^{1/3}}} \tag{43}$$

and

$$Nu_1 = \frac{(u_m^+/u_c^+)(T_c^+/T_m^+)Re(f/2)}{(1+\gamma)_{wmR^2}} \cong \frac{2a^+/u_m^+}{1+\dfrac{145}{(u_m^+)^{5/2}}} \tag{44}$$

Equations 43 and 44, with u_m^+ from Eq. 5, reproduce the computed values of Nu_0 and Nu_1 within 0.3%. These particular errors are further reduced in the predicted values of Nu itself for intermediate values of Pr/Pr_t. For specified values of Re, the corresponding value of a^+ may be determined by solving Eq. 5, with u_m^+ replaced by $Re/2a^+$, iteratively.

The values of Nu computed by Yu et al. and those predicted by the above procedures for specified values of Re and Pr share a common uncertainty, namely that associated with the prediction of Pr_t. Abbrecht and Churchill (1960) inferred, from their measurements of the velocity and two-dimensional (2-D) temperature field in fully developed turbulent flow in a round tube following a step in wall temperature, that Pr_t was a function only of Pr and the eddy viscosity, thereby independent of the mode of heating. From the close agreement of their values of Pr_t with those determined by Page et al. (1952) for heat transfer between parallel plates for $a^+ = b^+$, they inferred the independence of Pr_t from geometry as well. This sweeping generalization does not appear to have been either theoretically proven or disproven, but it is implied by the expressions derived by Yahkot et al. (1987) and Elperin et al. (1996) by renormalization group theory, as well as by most empirical correlating equations. (For example, see Kays (1994).)

Test calculations by Yu et al., using two different correlating equations for Pr_t, resulted in differences in Nu of less than 3% for all values of a^+ and y^+, thus supporting the qualitative deduction of Heng et al. (1998) that computed values of Nu are relatively insensitive to the expression used for Pr_t. Furthermore, these small differences appear to be associated with the dependence on Pr rather than on $\overline{(u'v')}^{++}$.

On the other hand, the Lagrangian direct numerical simulations of Papavassiliou and Hanratty (1997) imply that Pr_t becomes unbounded as $y^+ \to 0$ for very large values of Pr, which contradicts the basic postulate made in deriving Eq. 15. Even so, this does not necessarily invalidate Eqs. 41 and 42 insofar as the corresponding expression, such as that given by Shaw and Hanratty (1997), is used for Nu_∞.

The resolution of the uncertainty in Pr_t is the principal remaining task in turbulent forced convection. Equations 15, 43, and 44 may eventually need to be fine-tuned in this respect, but not necessarily Eqs. 41 and 42.

Assessment of the New Predictive Equations

The above emphasis on the errors and uncertainties associated with Eqs. 41 and 42, as well as with Eqs. 4, 5, 15, 43, and 44, should not be misconstrued. This general structure, together with its theoretically based components, is presumed to provide more accurate predictions of Nu for all Pr, all values of Re in the turbulent regime, all geometries, and all modes of heating than any previous expression or expressions. Furthermore, it provides a continuous functional relationship involving two points of inflection, of which the existence of one was apparently never recognized before.

Reassessment of the Functionality of Prior Theoretical and Empirical Equations

The presumed functional and numerical accuracy of the new correlative equations suggest a secondary role in addition to that of prediction, namely, their use as a standard for interpretation and evaluation of prior theoretical and empirical expressions for turbulent convection. The functionalities of the latter are considered in this section.

Prior analogies

Reynolds (1874) postulated that momentum and energy were transported between the bulk of the fluid stream and the confining surface wholly by the turbulent fluctuations in velocity and at equal mass rates. By elimination of that mass rate of turbulent transport, he derived the equivalent of the following

$$Nu = PrRe(f/2) \tag{45}$$

This expression shares with Eqs. 26 and 41 the twin distinctions of freedom from explicit empiricism and of complete generality with respect to geometry and the mode of heating at the surface.

Prandtl (1910) greatly improved on the Reynolds analogy by postulating linear transport of momentum and energy across a thin boundary layer of thickness δ, in series with the equal mass rate of transport of these quantities outside the boundary layer. On that basis, he derived the equivalent of

$$Nu = \frac{1}{\dfrac{1}{PrRe(f/2)} + \left(1 - \dfrac{1}{Pr}\right)\dfrac{\delta^+}{Re(f/2)^{1/2}}} \tag{46}$$

Equation 46 may be recognized as analogous to the reduced case of Eq. 26 for $Pr_t = 1$, Nu_1 from Eq. 14 with $\gamma = 0$, and Nu_∞ from Eq. 15 with δ^+ in place of $13.62Pr^{-1/3}$. It may be inferred from the Prandtl analogy that the Reynolds analogy is applicable only for $Pr = 1$, and, conversely, that the structure of the Reichardt analogy is simply a consequence of transport in series through the turbulent core and the viscous boundary layer, an inference that is not apparent from the derivation of Eq. 25. The failure of the Reynolds analogy for large values of Pr is, as indicated by comparison of Eqs. 45 and 15b, due to the failure to account for the presence of a viscous boundary layer, while failure of the Prandtl analogy in this same limit is due to the neglect of turbulent transport in the viscous boundary layer. The absence of γ in Eqs. 45 and 46 indicates that the deviation of the heat flux density distribution from the shear stress distribution within the fluid stream is neglected in both analogies. The common failure of the Reynolds, Prandtl, and Reichardt analogies for small values of Pr, as indicated by their comparison with Eq. 30 or 42, is due to their complete neglect of the contribution of thermal conduction within the turbulent core. The lasting merit of the Reynolds analogy is its prediction of the approximately correct dependence of Nu on both Re and Pr for $Pr = \mathcal{O}\{1\}$. The lasting merit of the Prandtl analogy is its prediction of a varying, interlinked dependence of Nu on Re and Pr. Unfortunately, even after nearly a century, this interlinking has been overlooked or ignored in the construction of many correlating equations.

Most of the many subsequently derived analogies for turbulent momentum and energy transfer may be categorized as empirical modifications of the Prandtl analogy. They generally incorporate a factor of $Pr^{1/3}$ to account for turbulent transport in the viscous boundary layer, but most imply that $Pr_t = 1$ and many that $\gamma = 0$. Since they are all inferior in detail to the corrected Reichardt analogy (Eq. 25) and the Churchill analogy (Eq. 41), and since they have all recently been reviewed by Churchill (1997c), they will not be examined here, except for two whose consideration is more conveniently deferred to a critique of empirical correlating equations.

Prior integral models

Many so-called analogies are actually approximate analytical or numerical solutions of the equivalent of Eq. 12 in terms of the eddy viscosity, utilizing separate and often incongruous expressions for the eddy viscosity and the time-averaged velocity. The friction factor then enters the expression for Nu only as a replacement for u_m^+. Particularly noteworthy examples are those of Martinelli (1947) and Lyon (1951), both of whom took thermal conduction in the turbulent core into account in order to provide better predictions for low-Prandtl-number fluids such as liquid metals. Both retained Pr_t in their formulations, but carried out numerical evaluations only for $Pr_t = 1$, thereby introducing significant error in the very range of Pr in which they were most interested. Lyon took into account the correct variation of the radial heat flux density, but Martinelli postulated $\gamma = 0$. Both neglected turbulent transport in the viscous boundary layer, thereby invalidating their results for very large values of Pr. Both noted the distinction between uniform and isothermal heating and postulated the former. Their major lasting contribution is the prediction of a lower bounding value for Nu as $Pr \to 0$.

Sleicher and Tribus (1957), Kays and Leung (1963), and Notter and Sleicher (1972) took all of these effects into account and thereby produced numerical solutions that are exact in principle for all values of Re and Pr. Their results have, however, now been superseded by those of Heng et al. (1998), Danov et al. (2000), and Yu et al. (2001), owing to the greater accuracy of Eq. 4 for $(\overline{u'v'})^{++}$ as compared to the earlier separate and incongruous expressions that were used for u^+ and μ_t. All six of these sets of numerical solutions are subject to the uncertainty still associated with the expression used for Pr_t.

Respresentative prior correlative equations

Nusselt (1910) misapplied dimensional analysis for forced convection and obtained

$$Nu = ARe^n Pr^m \tag{47}$$

rather than, correctly

$$Nu = \varphi\{Re, Pr\} \tag{48}$$

This mistake is deeply imbedded in the culture of chemical engineering. Equation 47 is apparently the original source of inspiration for the many correlating equations in the form of products of powers of the independent variables that pervade the literature, not only of heat and mass transfer, but also for many varieties of physical and chemical behavior. The Prandtl, Reichardt, and Churchill analogies totally refute this concept for the very behavior with which Nusselt was concerned. In reality, if more than one narrowly defined regime exists, power-dependences and their products occur only asymptotically. The numerical values of the arbitrary exponents found in the literature are usually some kind of mean for some particular range of the independent variable(s). The resulting functional, as well as numerical, misrepresentations are often disguised by the use of logarithmic coordinates for display. The serious consequence is the unwitting acceptance by designers of a relatively poor correlating equation whose mispredictions are necessarily compensated for by the use of a large safety factor. (It should be noted that the exponent of 2/3 in Eq. 41 has a theoretical basis, but that of 1/8 in Eq. 42 is purely empirical and presumably subject to the above criticisms.)

Dittus and Boelter (1930) correlated experimental data for forced convection in round tubes for both gases and ordinary liquids in terms of Eq. 47 with $n = 0.8$, $A = 0.0243$, and $m = 0.4$ for heating of the fluid, and with $n = 0.8$, $A = 0.0265$, and $m = 0.3$ for cooling of the fluid. These differences in m and Δ are now known to compensate for the effect of the variation in the viscosity of liquids with temperature. Colburn (1933) noted the similarity of the Dittus-Boelter equations to the following power-law-type empirical correlating equation for the friction factor

$$f = 0.046 Re^{-0.2} \tag{49}$$

Accordingly, he ingeniously postulated a mean value of 0.023 for the coefficient for both Nu and $f/2$ and a convenient mean value of 1/3 for the exponent of Pr, and then took the ratio of these two expressions to obtain

$$Nu = Re\left(\frac{f}{2}\right)Pr^{1/3} \tag{50}$$

Because of the absence of odd-valued exponents and a leading coefficient, and because of its degeneration to the

Reynolds analogy for $Pr = 1$, the Colburn analogy is sometimes misinferred to have some theoretical basis. However, the absence of an odd-valued exponent for Re and a leading coefficient were simply contrived, and the degeneration of Eq. 50 to Eq. 45 is simply fortuitous. Actually, as may be inferred from Eq. 5, f is not a fixed power of Re for any extended range of Re. Also, as may be inferred from Eqs. 41 and 42, Nu is not proportional to a fixed power of Re or Pr for any extended range of either variable, even for $Pr > Pr_t$. In summary, Eq. 50, which is a combination of two arbitrary empirical correlating equations, is in error functionally in every respect. As will be shown, it also provides poorer numerical predictions than might have been expected on the basis of its empirical roots.

Many improved correlating equations have been proposed since the time of Dittus and Boelter and of Colburn. Most of them have the general structure of the Prandtl, Reichardt, or Churchill analogies, but incorporate empirical coefficients. Since most of these expressions are grossly inferior to Eqs. 41 and 42 with the proposed auxiliary expressions for Nu_0, Nu_1, and Nu_∞, and since they generally do not even purport to encompass the regime of small values of Pr, they will not be examined here except for one based on Eq. 23, one based on the structure of Eq. 41, and one based on Eq. 21.

Friend and Metzner (1958) rearranged the equivalent of Eq. 24 as follows in terms of μ_t and for the very simplified case of $\gamma = 0$, $Pr_t = 1$, and $T_c^+ = T_m^+$

$$\int_0^{u_c^+} \frac{du^+}{1 + Pr\left(\dfrac{\mu_t}{\mu}\right)} = \frac{\dfrac{PrRe(f/2)}{Nu} - \dfrac{u_c^+}{u_m^+}}{(Pr-1)(f/2)^{1/2}} \tag{51}$$

Using the following correlating equation of Drew et al. (1932) for the friction factor

$$\frac{f}{2} = 0.0007 + 0.0625Re^{-0.32} \tag{52}$$

and experimental data for Nu (and Sh) for various values of Re and Pr (and Sc) ≥ 0.7, they determined a mean value of 1.2 for u_c^+/u_m^+, and $11.8Pr^{-1/3}$ as a representation for the integral. Their final result was therefore

$$Nu = \frac{1}{\dfrac{1.2}{PrRe(f/2)} + \left(1 - \dfrac{1}{Pr}\right)\dfrac{11.8}{Re(f/2)^{1/2}Pr^{1/3}}} \tag{53}$$

Equation 53 is found, as might be expected, to represent reasonably well the experimental data upon which it was based. Comparison of this expression with Eqs. 25 and 26 indicates that $Pr_t(T_m^+/T_c^+)(u_c^+/u_m^+)(1+\gamma)_{mu^+}$ has been approximated by 1.2 and $13.62[1-(Pr_t/Pr)](T_m^+/T_c^+)Pr^{1/3}$ by $11.8\,(1-1/Pr)$, or Nu_1 by $[PrRe(f/2)/1.2]$ and Nu_∞ by

$$\left(\frac{Pr - Pr_t}{Pr - 1}\right)\frac{Re(f/2)^{1/2}Pr^{1/3}}{11.8}$$

Friend and Metzner noted a point of inflection at $Pr \cong 2$ in their logarithmic plots of $Nu/PrRe$ vs. Pr, but dismissed that observation with the statement, "This representation has no theoretical significance, of course; it is merely a consequence of the simple empirical from chosen (to represent the integral of Eq. 54) and the approximate treatment of the second-order effects of T_c^+/T_m^+ and u_c^+/u_m^+, ..."

Petukhov (1970) proposed the correlating equation

$$Nu = \frac{1}{\dfrac{K_1}{PrRe(f/2)} + \dfrac{K_2[Pr^{2/3} - 1]}{PrRe(f/2)^{1/2}}} \tag{54}$$

with

$$K_1 = 1 + 3.4(f/2) \cong 1.07 \tag{55}$$

and

$$K_2 = 11.7 + 1.8Pr^{-1/3} \cong 12.8 \tag{56}$$

to represent the numerically computed values of Nu by Petukhov and Popov (1963) for $Pr \geq 0.5$ and uniform heating. In evaluating these coefficients, they utilized the following expression for the friction factor

$$\left(\frac{2}{f}\right)^{1/2} = 2.235\ln\{Re\} - 4.64 \tag{57}$$

Equation 54 may be inferred to be equivalent to Eq. 41 with $Nu_1 = Pr_t Re(f/2)K_1$ and

$$Nu_\infty = \frac{Re(f/2)^{1/2}Pr^{1/3}\left[1 - (Pr_t/Pr)^{2/3}\right]}{K_2\left[1 - \dfrac{1}{Pr^{2/3}}\right]}$$

that is with $(1+\gamma)_{wmR} = Pr_t(T_c^+/T_m^+)(u_m^+/u_c^+)$ approximated by K_1, and

$$13.62\left[1 - \left(\frac{Pr_t}{Pr}\right)^{2/3}\right] \Big/ \left[1 - \frac{1}{Pr^{2/3}}\right]$$

by K_2.

Churchill (1977) developed the following correlating equation for all values of Re and Pr for both a uniformly heated and an isothermally heated round tube

$$Nu = Nu_0 + \frac{0.079PrRe(f/2)^{1/2}}{(1 + Pr^{4/5})^{5/6}} \tag{58}$$

As contrasted with Eqs. 53 and 54, Eq. 58 is based on the application of Eq. 21 for an expression for large Pr, an expression for intermediate Pr, and a limiting value for $Pr = 0$, with $p = 1$ and $q = -5/6$. The coefficient of 0.079, the implicit coefficient of unity before $Pr^{4/5}$, the combining exponents, and the limiting values of 6.3 and 4.8 for uniform heating and isothermal heating, respectively, were based on culled experimental data for both heat and mass transfer, as well as on the computed values of Notter and Sleicher (1972). A complementary expression differing only slightly from Eq. 5

was devised for the friction factor. Equation 58 is in error functionally in that it predicts only one point of inflection with Pr, does not predict the proportionality of Nu to $Re(f/2)$ for $Pr = Pr_t$, and thereby does not predict the interlinking of the dependence of Nu on Re and Pr.

Graphical Comparison of the Accuracy of the Numerical Predictions of the Past and Present

In the preceding section it was demonstrated that Eqs. 41 and 42, together with Eqs. 5, 13–15, 18, and 19, constitute a significant advance in functionality, scope, and generality over all prior analyses. In this section, the numerical predictions of the new expressions, as well as those of the past, are compared with essentially exact numerically computed values. This comparison is limited to isothermal round tubes, as are most of the detailed derivations in this article, but limited test calculations suggest that the results are representative for uniformly heated round tubes, for parallel plates, and presumably for other geometries as well. In order to emphasize the magnitude of the differences, the comparisons are presented in terms of the percentage deviations from the computed values of Yu et al. (2001). The only significant source of uncertainty in these latter values is that associated with the expression used to predict Pr_t, namely Eq. 32, since the uncertainty in Nu associated with the uncertainty of Eqs. 4 and 5 is truly negligible. Except for the correlating equation of Petukhov (1970), the comparisons are for uniform wall temperature. In that one case, the comparison is for uniform heating in accordance with their correlated values. The analogies of Colburn and of Friend and Metzner were based on experimental data for which the mode of heating was not clearly or consistently defined in all cases. In any event, the differences due to the mode of heating are negligible for their ranges of values of Pr. The expressions used by individual authors for the friction factor were utilized with their expressions for Nu. This shifts the Reynolds number somewhat for a fixed value of a^+, but neither that shift nor the effect of different expressions for f in the predictive expressions for Nu significantly affects the comparisons that follow.

The percent deviations of the various expressions for Nu from the computed values of Yu et al. are plotted in Figure 3 for $a^+ = 5{,}000$ ($Re = 2.27 \times 10^5$). The deviations due to Eq. 41 are seen to be completely negligible for $Pr > Pr_t$, even in this exaggerated form, and to be very small, with a maximum value of $\sim 1.5\%$, for $Pr < Pr_t$. The seemingly random deviations for $Pr < Pr_t$ suggest the possibility that some of these discrepancies may be due to small numerical errors in the computations rather than to the predictive equations. The accuracy of the predictions of Eq. 41 allows this expression to be used as a standard for values of Pr intermediate to those chosen by Yu et al. for their computations, resulting in completely defined curves for the graphical comparisons for $Pr > Pr_t$. However, for $Pr < Pr_t$, the curves were necessarily sketched through the values of the deviations for only five discrete values of Pr.

The Colburn analogy (Eq. 50), which is effectively the Dittus-Boelter equation in this instance, may be observed to overpredict Nu by 8% at $Pr = 0.7$ and then to underpredict increasingly to 43% as Pr increases to 10,000. For $a^+ = 50{,}000$ (not shown in Figure 3), the overprediction of Eq. 50 at $Pr =$ 0.7 increases slightly to 9% and the underprediction at $Pr =$ 10,000 decreases to 26%. At $a^+ = 500$, the overprediction of Eq. 50 at $Pr = 0.7$ decreases to 2.4%, but the underprediction at $Pr = 10{,}000$ increases to 56%. It may be concluded from these results that, at least for channels, the Colburn analogy is now only of historical interest.

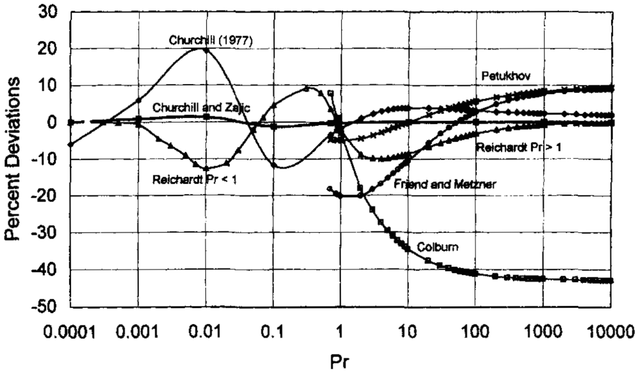

Figure 3. Percent deviations of predictions of various correlative expressions from calculated values of Yu et al. (2001) for round tubes with uniform heating or a uniform wall temperature at $a^+ = 5{,}000$.

The analogy of Friend and Metzner (Eq. 53) may be observed in Figure 3 to underpredict Nu by $\sim 20\%$ for $Pr = \mathcal{O}\{1\}$ and overpredict by $\sim 9\%$ for large values of Pr. In contrast to the analogy of Colburn, the deviations do not change greatly with a^+. A detailed analysis suggests that these deviations are primarily a consequence of nonoptimal values for the two arbitrary coefficients of Eq. 53, but also to some extent of the inferiority of the dependence on the factor $1-(1/Pr)$ as compared to $1-(Pr_t/Pr)^{2/3}$.

Equation 54 of Petukhov may be observed to underpredict somewhat less than Eq. 53 of Friend and Metzner ($\sim 10\%$) for $Pr = \mathcal{O}\{1\}$, but to overpredict almost identically to Eq. 53 for large values of Pr. The deviations for $Pr > Pr_t$ are primarily a consequence of less than optimal expressions for K_1 and K_2 since Eq. 54 is otherwise identical to Eq. 41. The neglect of γ is an additional significant source of error for the lower values of Pr.

Equation 25, the corrected analogy of Reichardt (1951) which provided the basis for many of the developments herein, underpredicts Nu by about 10% at $Pr = 3$, but is exact at $Pr = Pr_t = 0.8673$ and for $Pr \to \infty$.

Equation 58 of Churchill (1977) may be observed to be superior numerically to all but Eq. 41 for $Pr > Pr_t$, but to mispredict the computed values somewhat randomly and by as much as 20% for $Pr < Pr_t$. It is even inferior to Eq. 31 in this latter regime. The predictions of Eq. 58 are even poorer for $a^+ = 500$ and 50,000 for small values of Pr.

All in all, Figure 3 constitutes a testimonial to the significantly improved predictions provided by Eqs. 41 and 42 over all prior expressions.

Validity, Empiricism, and Uncertainty

This section attempts to assess Eqs. 41 and 42, together with their components Nu_0, Nu_1, Nu_∞, and Pr_t/Pr both qualitatively and quantitatively in terms of their validity, their empirical content, and the consequent uncertainty of their predictions.

Equation 41 is free of any explicit empiricism, but is based on the analogy of Churchill (1997a) between energy and momentum transfer, which incorporates in its derivation some conjecture, including the specific behavior represented by Eq. 19 for $Pr = Pr_t$ and the asymptotic behavior represented by Eq. 15 for $Pr \to \infty$.

Equation 42 is purely conjectural and incorporates one purely empirical exponent, namely 1/8 for Pr_t/Pr. Its justification is analogic and pragmatic.

Equations 13 and 18 for Nu_0 are exact unambiguously, but their numerical evaluation invokes empiricism by virtue of the correlating equation for $(\overline{u'v'})^{++}$, namely Eq. 4.

Equations 14 and 19 for Nu_1 are exact insofar as a value of Pr exists for which Pr_t is independent of y^+ and a^+. This somewhat surprising aspect of behavior is actually predicted by renormalization group theory [see, for example, Yahkot et al. (1987) and Elperin et al. (1996)] for the turbulent core, and is supported by most experimental data for the region near the wall. The value of Pr for which Pr_t is independent of y^+ and a^+ is perhaps uncertain within the range of 0.79 to 0.87, but the effect of this uncertainty on the prediction of Nu is negligible. The numerical evaluation of Nu_1 also evokes empiricism by virtue of the correlating equation for $(\overline{u'v'})^{++}$, namely Eq. 4.

The empiricism of Eq. 4 is represented by the coefficients 0.436 and 6.95 and the exponent $-8/7$. Although the consequent uncertainty in the predictions of $(\overline{u'v'})^{++}$ may be as much as 5%, the consequent uncertainty in $1+\gamma$, u_m^+, and u_c^+, and in turn in Nu_0 and Nu_1, is probably less than 1% as a result of the smoothing provided by the intervening integrations or double integrations.

The uncertainty in the correlative expressions for Nu_0 and Nu_1, namely Eqs. 43 and 44, or their counterparts for other geometries and modes of heating, is less than 0.3%. Equations 43 and 44, and, thereby, even this small error, can be avoided by calculating Nu_0 and Nu_1 by means of Eqs. 18 and 19 or the equivalent.

The validity of the various forms of Eq. 15 for $Pr \to \infty$ depends on the existence of a finite limiting value for Pr_t as $y^+ \to 0$, and their accuracy depends on the 1/3-power of that limiting value. Experimental data for Nu as well as for Pr_t indicate a value in the range of 0.79 to 0.90. The corresponding uncertainty in Nu_∞ is perhaps 2%. The Lagrangian and Eulerian direct numerical simulations of Papavassiliou and Hanratty (1997) suggest that a limiting value for Pr_t may not exist for $Pr > 100$. Pr is less than 100 for all ordinary fluids, but these findings raise doubt as to the applicability of Eq. 41 for mass transfer, for which values of Sc much greater than 100 may be encountered. Even so, Eq. 41 remains applicable if the appropriate value is utilized for Nu_∞ rather than that predicted by Eq. 15.

The uncertainty in Pr_t is greatly dampened in the prediction of Nu by Eqs. 41 and 42, as reasoned by Heng et al. (1998) and demonstrated by Yu et al. (2001), who compared the predictions of Nu for two radically different expressions for Pr_t.

All in all, the uncertainty in the values of Nu as predicted by Eqs. 41 and 42 due to implicit empiricism is probably less than 3% and most likely of the order of 1%. The assertion of minimal empiricism is presumably justified in that sense. This remarkable achievement in a subject of such basic complexity is a consequence of the exploitation of analogies and asymptotes, as well as fortuitously to the dampening of uncertainties by integrations.

Summary and Conclusions

Churchill et al. (2000) deduced a generic correlating equation free of any explicit empiricism based on the venerable analogy of Reichardt (1951). This expression only has validity for $Pr \geq Pr_t$, but they devised a compatible analogue for $Pr \leq Pr_t$ by conjecture. They speculated that these two generic expressions might be applicable for all geometries and modes of heating, and confirmed this speculation graphically using the essentially exact computed values of Heng et al. (1998) for a uniformly heated round tube, and of Danov et al. (2000) for parallel plates with two different modes of heating. This result was subsequently reconfirmed by the even more accurately computed values of Yu et al. (2000) for a round tube with both isothermal and uniform heating. In the current investigation these representations were examined more critically and in greater detail. Despite the good overall agreement found by Churchill et al. (2000), significant discrepancies were discovered for some particular values of Pr. Those for $Pr > Pr_t$ were shown to be due to the idealizations made by Reichardt in order to be able to integrate analytically. It is apparent that freedom from empiricism does not guarantee exactness. The Reynolds analogy is another example of this truism.

A greatly improved expression for $Pr > Pr_t$ with the same generality and freedom from empiricism was derived on the basis of an analogy of Churchill (1997a). Again, a compatible analogue for $Pr < Pr_t$ was devised by conjecture, however, with the introduction of some empiricism as well this time. The two new expressions represent the essentially exact computed values for all $a^+ > 150$ within about 0.25% for $Pr > Pr_t$ and within about 1.0% for $Pr < Pr_t$. For all practical purposes, the predictions for a specified value of Pr_t/Pr may be considered exact.

By virtue of the recognition of the similarity of the basic structure of the Reichardt and Churchill analogies to the staggered canonical correlating equation of Churchill and Usagi (1972) for three regimes, the new expressions were discovered to predict two sigmoidal transitions as Pr increases. At least one such transition was predictable in principle from many of the analogies of the past, but the second was apparently never recognized because of the use of insensitive graphical forms for display.

These new expressions were shown on the basis of a comparison with essentially exact computed values to be superior both functionally and numerically to the many analogies and correlating equations of the past.

Comparison of the final generic expressions in terms of Pr_t/Pr with experimental data is unessential and inappropriate since they are essentially exact. The only source of empiricism, even in Nu_0, Nu_1, and Nu_∞, arises from the correlating equation for the turbulent shear stress, which is based on experimental data to some extent, but even so, results in almost no uncertainty in Nu.

Some empiricism is required to predict values of Nu for specified values of Pr rather than of Pr/Pr_t, namely an expression for Pr_t. A general and well-accepted theoretical or empirical expression for this quantity does not yet exist, but, fortunately, the predicted values of Nu are relatively insensitive to Pr_t and the choice of a tentative predictive expression thereof. The development of a reliable expression for Pr_t is the principal remaining task in turbulent convection.

Comparison of predicted values of Nu for specified values of Pr with experimental data is not productive since the dispersion in even the best experimental data owing to the variation of physical properties with temperature, incomplete thermal development, and undefined thermal boundary conditions, is greater than the uncertainty in the predictions associated with Pr_t. (See, for example, Churchill (1977).)

The success of Eqs. 41 and 42 for such complex behavior as turbulent convection suggests the potential of developing equivalent theoretically based expressions for other aspects of physical, chemical, and biological behavior. The long chain of development of these final two expressions for $Nu\{Pr_t/Pr, a^+\}$ provides guidance and encouragement for such applications. The blunt presentation of these final results without the preceding analogies would be quite misleading in that respect. For example, the potential generality of the Reichardt analogy would not have been recognized without a prior familiarity with the theoretical expressions for Nu_1 and Nu_∞, nor its extension for $Pr < Pr_t$ without that for Nu_0. Without the generalization of the Reichardt analogy, the simplification and generalization of the Churchill analogy would not have occurred even to the current authors. Without a recognition of the similarity of the structure of Eqs. 22 and 27, Eq. 30 would not have been derived, and the surprisingly simple functional character of the dependence of Nu on Pr in Eqs. 30 and 42 for $Pr < Pr_t$ would not have been deduced.

Finally, Eqs. 41 and 42, together with Eq. 15c, and with Eqs. 43 and 44 or their equivalents for other geometries and modes of heating, appear to be the most accurate and comprehensive correlative equations for turbulent forced convection in channels and possibly for unconfined flows as well. As such, they appear to be worthy of adoption in our textbooks, handbooks, and computer codes in place of the traditional ones examined in Figure 3. Equation 32 is recommended tentatively for the prediction of Pr_t, with the expectation that improved expressions will be devised in the near future. Equation 5 and its analogues for other geometries are recommended for the calculation of Re for a specified value of a^+ or the equivalent.

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The final version of this manuscript was greatly improved by the constructive criticisms and suggestions of the reviewers. In response to their concern for the implied criticisms of the early work by Prandtl, Colburn, and so on, the perhaps appropriate response is that of Newton who said, "If I have seen further, it is by standing on the sholders [*sic*] of Giants."

Notation

a = radius of tube
$a^+ = a(\tau_w \rho)^{1/2}/\mu$
b = half-spacing between parallel plates
A = arbitrary coefficient in Eq. 47
b = half-spacing between parallel plates
$b^+ = b(\tau_w \rho)^{1/2}/\mu$
c_p = specific heat capacity at constant pressure
$f = 2\tau_w/\rho u_m^2$; Fanning fraction factor
j = heat flux density in y-direction
k = thermal conductivity
m = arbitrary exponent of Pr in Eq. 47
n = arbitrary exponent of Re in Eq. 47
$Nu = 2aj_w/k(T_w - T_m)$; Nusselt number for round tube
$Nu_0 = Nu\{Pr = 0\}$
$Nu_1 = Nu\{Pr = Pr_t\}$
$Nu_\infty = Nu\{Pr \to \infty\}$
$Nu_\infty^1 = Nu_\infty\{Pr = Pr_t\}$
Nu_i = intermediate asymptote
$\mathcal{O}\{x\}$ = of order of magnitude x
p = arbitrary exponent in Eq. 20
$Pr = c_p \mu/k$; Prandtl number
$Pr_t = Pr(\overline{u'v'})^{++}[1-(\overline{T'v'})^{++}]/(\overline{T'v'})^{++}[1-(\overline{u'v'})^{++}]$
q = arbitrary exponent in Eq. 21
$R = 1 - y/a$
$Re = 2au_m \rho/\mu$; Reynolds number for round tube
Sc = Schmidt number
T = time-averaged temperature
$T^+ = k(\tau_w \rho)^{1/2}(T_w - T)/\mu j_w$
T' = fluctuating component of temperature
$\overline{T'v'}$ = time-averaged value of $T'v'$
$(\overline{T'v'})^{++} = \rho c_p \overline{T'v'}/j$
u = time-averaged axial component of velocity
$u^+ = u(\rho/\tau_w)^{1/2}$
u' = axial component of velocity
$\overline{u'v'}$ = time-averaged value of $u'v'$
$(\overline{u'v'})^{++} = -\rho\overline{u'v'}/\tau$
v' = fluctuation of y-component of velocity
x = axial coordinate
x_0 = point of onset of heating
y = distance from wall
$y\{z\}$ = arbitrary function of arbitrary variable
$y_0\{z\} = y\{z \to 0\}$
$y_\infty\{z\} = y\{z \to \infty\}$
$y_i\{z\}$ = intermediate asymptote for $y\{z\}$
z = arbitrary variable

Greek letters

α = arbitrary coefficient in Eq. 29
$\gamma = (j/j_w)(\tau_w/\tau) - 1$
δ = thickness of viscous boundary layer
$\delta^+ = \delta(\tau_w \rho)^{1/2}/\mu$
μ = dynamic viscosity
μ_t = dynamic eddy viscosity
ρ = specific density
τ = shear stress
$\varphi\{z\}$ = arbitrary function of arbitrary variable
$\Psi\{Pr_t/Pr\}$ = arbitrary function of Pr_t/Pr

Subscripts

c = at centerline
m = space-mean value (for temperature, weighted by u)
w = at wall
mR^2 = integrated-mean with respect to R^2
mR^4 = integrated-mean with respect to R^4
mu^+ = integrated-mean with respect to u^+
wmR^2 = integrated-mean, weighted by $[-(\overline{u'v'})^{++}]$, with respect to R^2
wmR^4 = integrated-mean, weighted by $[-(\overline{u'v'})^{++}]$, with respect to R^4

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